Solutions Exam Signals and Systems

23 januari 2014, 18:30-21:30

Problem 1: signals and spectra

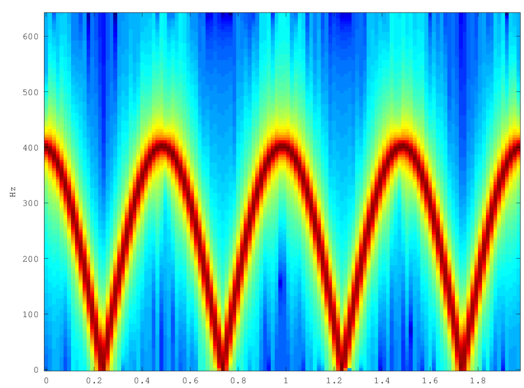
(a) The amplitude of the first signal is clearly 2. Its period is 2 seconds, so its frequency is 0.5Hz. Its delay relative to a standard cosine is -1.5π, which corresponds with a phase angle of $\pi/2$. We conclude $x(t) = 2\cos(\pi t + \pi/2)$. Similarly, the second signal has amplitude 1, frequency 2Hz and phase $\phi = \pi$, so $y(t) = \cos(4\pi t + \pi)$. Careful inspection of the third plot shows that it is an AM signal that is constructed from the other two plots: $z(t) = x(t)y(t) = 2\cos(\pi t + \pi/2)\cos(4\pi t + \pi)$.

(b) We use the inverse Euler formula $cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2}$.

$$\begin{aligned} x(t) &= 4\sin(\pi 10t) = 4\cos(\pi 10t - \pi/2) = 2e^{-j\pi/2}e^{j\pi 10t} + 2e^{j\pi/2}e^{-j\pi 10t} \\ y(t) &= \cos(\pi 4t + \pi/4) = \frac{1}{2}e^{j\pi/4}e^{j\pi 4t} + \frac{1}{2}e^{-j\pi/4}e^{-j\pi 4t} \\ z(t) &= x(t)y(t) = e^{-j\pi/4}e^{j\pi 14t} + e^{j\pi/4}e^{-j\pi 14t} + e^{-j3\pi/4}e^{j\pi 6t} + e^{j3\pi/4}e^{-j\pi 6t} \end{aligned}$$

(c) These plots can be made using the answers from part (b). Make sure that you put labels at the axis (so f in Hz, or rad/s). Also, you need to specify for each frequency component the corresponding phase angle.

(d) When you listen to the signal $x(t) = 10\cos(800\pi\sin(2\pi t))$ you will hear a frequency oscillating from 400Hz to 0Hz. You will hear something like a siren. The spectrogram is in the following figure.



Problem 2: LTI-systems

(a) Note that $\cos\left(2\pi(n+1) - \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ since n is integer. So,

$$\begin{aligned} y_0[n] &= x[n]\cdot x[n-1] + \cos\left(\frac{\pi}{3}\right) \\ y_1[n] &= x[n-1] + x[n] + x[n-1] \end{aligned}$$

Now we see that both systems are causal, since they use only sample locations of the type $n - k$, where k a natural number. The first system is not linear due to $x[n] \cdot x[n-1]$, while the second is linear (in fact, it is a standard FIR filter). Both signals are time invariant.

(b) The impulse response is the output of feeding the system with $\delta[n] = u[n] - u[n-1]$, so the answer is simply (using linearity of FIR systems):

$$[3, 4, 8, 9, 14] - [0, 3, 4, 8, 9] = [3, 1, 4, 1, 5]$$

(c) The convolution of $[1, -1, -2]$ and $[0, 4, 2, 0, -4, -2]$ yields $[0, 4, -2, -10, -8, 2, 10, 4]$.

(d) There are many ways to answer this question. Probably the simplest is the following reasoning. If $y = h * x$, then $|y| = |h| + |x| - 1$, so if we want the filter kernel h_4 to undo the work of h_3, then it needs to have a negative length, which is impossible.

(e) Using the reasoning that $|y| = |h| + |x| - 1$, we know that $|h| = 5$. We compute the convolution of $[a, b, c, d, e]$ with x and find:

$$[a, b, c, d, e] * [1, -1, -2] = [a, b - a, c - b - 2a, d - c - 2b, e - d - 2c, -e - 2d, -2e]$$

Since this must be equal to $[0, 3, -2, -2, -5, -12, -4]$ we conclude $a = 0$, $b = 3$, $c = 1$, $d = 5$, and $e = 2$. So, $h_5 = [0, 3, 1, 5, 2]$.

(f) It is indeed possible to build the FIR-system F_3 by connecting the first four elements, since

$$[0, 4, 2, 0, -4, -2] = [0, 1] * [1, -1] * [2, 2, 2] * [2, 1] = c_0 * c_1 * c_2 * c_3$$

Problem 3: z-transforms

(a) $H(z) = 1 - 2\cos(\hat{\omega})z^{-1} + z^{-2}$ and $H(e^{j\omega}) = 1 - 2\cos(\hat{\omega})e^{-j\omega} + e^{-j2\omega}$

(b) The signal $x[n] = A\cos(n\hat{\omega} + \phi)$ will result in an output which is zero everywhere if $H(e^{j\hat{\omega}}) = 0$, which is indeed the case:

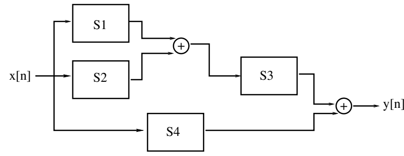
$$1 - 2\cos(\hat{\omega})e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} = e^{-j\hat{\omega}}(e^{j\hat{\omega}} - 2\cos(\hat{\omega}) + e^{-j\hat{\omega}}) = e^{-j\hat{\omega}}(2\cos(\hat{\omega}) - 2\cos(\hat{\omega})) = 0$$

(c) We use linearity, and treat the terms separately. The DC-term is removed by a first difference, i.e. $h_0 = [1, -1]$. From part (b) we know that the other two filters are $h_1 = [1, -2\cos(\frac{\pi}{3}), 1] = [1, \frac{1}{2}, 1]$ and $h_2 = [1, -2\cos(\frac{\pi}{4}), 1] = [1, \frac{1}{2}\sqrt{(2)}, 1]$. The total system is

$$h = h_0 * h_1 * h2 = [1, -1] * [1, \frac{1}{2}, 1] * [1, \frac{1}{2}\sqrt{(2)}, 1]$$

Another solution is the following. Choose $h_1 = [1, 1, 1, 1, 1, 1]$ (a six-point averager) and $h_2 = [1, 1, 1, 1, 1, 1, 1, 1]$ (a 8-point averager). Again, the answer is $h = h_0 * h_1 * h2$. This filter works as well, but is less specific.

(d) We consider a connected system that is composed of four FIR systems S_1, S_2, S_3 and S_4.



The impulse responses are $h_1 = [1, 1]$, $h_2 = [1, 0, -1]$, $h_3 = [0, 0, 0, 2]$ and $h_4 = [0, 1, 0, 1]$. The overall system (using linearity) is thus:

$$h = (h_1 + h_2) * h_3 + h_4 = [0, 1, 0, 5, 2, -2]$$

So, we find

- system function: $H(z) = z^{-1} + 5z^{-3} + 2z^{-4} - 2z^{-5}$
- frequency response:$H(e^{j\omega}) = e^{-j\omega} + 5e^{-j3\omega} + 2e^{-j4\omega} - 2e^{-j5\omega}$
- impulse response: $h[n] = \delta[n-1] + 5\delta[n-3] + 2\delta[n-4] - 2\delta[n-5]$
- difference equation: $y[n] = x[n-1] + 5x[n-3] + 2x[n=4] - 2x[n-5]$

Problem 4: Fourier analysis

(a) According to the Fourier synthesis formula (using $T_0 = 1/2$):

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j4\pi kt} = 4e^{-12\pi t} + 2e^{-j\pi/4}e^{-j4\pi t} + 5e^0 + 2e^{-j\pi/4}e^{j4\pi t} + 4e^{-12\pi t} \\ &= 5 + 4\cos(2\pi 2t + \pi/4) + 8\cos(2\pi 6t) \end{aligned}$$

So, $DC = 5$, $A = 4$, $f_0 = 2$, $\phi_0 = \pi/4$, $B = 8$, $f_1 = 6$, and $\phi_0 = 0$.

(b) We can read the Fourier coefficients directly from the formula. However, we first need to determine the fundamental frequency $f_0 = \gcd(6, 9) = 3$Hz. So, the cases are $k = 0$, $k = \pm 2$ and $k = \pm 3$.

$$a_k = \begin{cases} \frac{3}{2}e^{-j\pi/2} & \text{for } k = -3 \\ 1 & \text{for } k \in \{-2, 0, 2\} \\ \frac{3}{2}e^{j\pi/2} & \text{for } k = 3 \\ 0 & \text{for all other } k \end{cases}$$

(c) According to the Fourier analysis formula we find:

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}\,dt = \frac{1}{T_0}\int_{aT_0}^{bT_0} e^{-j(2\pi/T_0)kt}\,dt - \frac{1}{T_0}\int_{cT_0}^{dT_0} e^{-j(2\pi/T_0)kt}\,dt$$

For the DC -term (i.e. $k = 0$ and hence $e^{-j(2\pi/T_0)kt} = 1$) this reduces to:

$$a_0 = \frac{1}{T_0}\int_{aT_0}^{bT_0} 1\,dt - \frac{1}{T_0}\int_{cT_0}^{dT_0} 1\,dt = \frac{1}{T_0}\left(bT_0 - aT_0 - d_T 0 + cT_0\right) = b + c - a - d$$

For other k we find:

$$a_k = \frac{1}{T_0}\left[\frac{e^{-j(2\pi/T_0)kt}}{-j(2\pi/T_0)k}\right]_{t=aT_0}^{t=bT_0} - \frac{1}{T_0}\left[\frac{e^{-j(2\pi/T_0)kt}}{-j(2\pi/T_0)k}\right]_{t=dT_0}^{t=cT_0}$$

Substitution and some trivial calculus leads to:

$$a_k = \frac{-1}{j2\pi k}\left(e^{-j2\pi kb} - e^{-j2\pi ka} + e^{-j2\pi kc} - e^{-j2\pi kd}\right)$$

(d) We can use the answer from (c), using $a = 0$, $b = c = \frac{1}{2}$ and $d = 1$. We thus arrive at $a_0 = b + c - a - d = 0$ and (for k$\neq 0$) $a_k = \frac{-1}{j2\pi k}\left(e^{-j\pi k} - e^0 + e^{-j\pi k} - e^{-j2\pi k}\right) = \frac{-1}{j2\pi k}\left(2(-1)^k - 2\right)$ For k even, this is equal to zero, while for k odd we find $a_k = \frac{-1}{j2\pi k}\left(2(-1) - 2\right) = \frac{2}{j\pi k}$.

(e) The periodic signal $y(t)$ (with period 6 sec.) can be rewritten as the sum of the signals $x(t)$ and $s(t)$, where we choose in the formula for $s(t)$ the parameters $a = \frac{1}{6}$, $b = \frac{2}{6}$, $c = \frac{4}{6}$, and $d = \frac{5}{6}$. The Fourier coefficients of the signal $y(t)$ can therefore (using linearity) be found by adding the answer found by (c) and (d). So, $a_0 = \frac{2+4-1-5}{6} = 0$ and (for $k \neq 0$)

$$a_k = \frac{-1}{j2\pi k}\left(2(-1)^k - 2 + e^{-j\frac{2}{3}\pi k} - e^{-j\frac{1}{3}\pi k} + e^{-j\frac{4}{3}\pi k} - e^{-j\frac{5}{3}\pi k}\right)$$